This week

1. Lines in $\mathbb{R}^2$
2. Section 12.5: lines and planes in space
3. Application: perspective projection
Points and vectors

**Convention**

- *From now on we will identify points with terminal points of vectors in standard position:*
  - $P = \mathbf{v}$
- *We will abandon the notation $\langle x_1, \ldots, x_n \rangle$ and use $(x_1, \ldots, x_n)$ instead.*

**Parametrisation**

**Definition**

*A line in $\mathbb{R}^2$ is defined by an equation of the form*

$$\ell: ax + by = c \quad (*)$$

*with $a$, $b$ and $c$ real numbers.*

- The line $\ell$ consists of the points that satisfy equation $(*)$:
  $$\ell = \{(x, y) \mid ax + by = c\}.$$
- The line $\ell$ is the **solution set** of equation $(*)$. 
**Definition**

A parametrisation of the line $\ell$ is a function $r: \mathbb{R} \rightarrow \mathbb{R}^2$ such that $r(t)$ reaches all points of $\ell$ while $t$ runs through all real numbers.

- The number $t$ is called the **parameter**.
- The line $\ell$ is the set of all points $r(t)$:
  \[ \ell = \{ r(t) \mid t \in \mathbb{R} \} . \]
- The function $r(t)$ has two components that both depend on $t$:
  \[ r(t) = (x(t), y(t)) . \]
- Functions like $r$ with values in $\mathbb{R}^n$ are called **vector functions**.

**Example**

Given is the line $\ell$: $2x + 3y = 6$. Find a parametrisation of $\ell$. 

\[ \ell = \{ (x, y) \mid 2x + 3y = 6 \} . \]
Example

Find an equation for the line

\[ \ell: (3t, 2 - 2t), \quad t \in \mathbb{R}. \]

Theorem

For every line \( \ell \) there exist numbers \( p_1, p_2, v_1 \) and \( v_2 \) such that

\[ r(t) = (p_1 + v_1 t, p_2 + v_2 t) \quad t \in \mathbb{R}. \]

- Write \( r(t) \) as follows:
  \[ r(t) = (p_1, p_2) + t(v_1, v_2). \]
- The vector \( p = (p_1, p_2) \) is called a support vector of \( \ell \).
- The vector \( v = (v_1, v_2) \) is called a direction vector of \( \ell \).
- Define \( q = r(1) \), then
  \[ r(1) = p + v, \quad \text{dus} \quad v = q - p. \]
- The parametrised vector form of \( \ell \) is
  \[ \ell: r(t) = p + t v \quad t \in \mathbb{R}. \]
Example

Find a support- and a direction vector of the line \( \ell: 2x + 3y = 6 \), and find a parametrised vector form of \( \ell \).

Lines in space

Definition

Let \( p \) and \( v \neq 0 \) be vectors. The \textbf{parametrised vector form} of the line through \( p \) and parallel to \( v \) is

\[
\mathbf{r}(t) = p + tv, \quad t \in \mathbb{R}.
\]

- The vector \( p \) is called a \textbf{support vector} and the vector \( v \) is called a \textbf{direction vector} of the line.
- If \( \mathbf{r}(t) = (f(t), g(t), h(t)) \), then the equations

\[
\begin{align*}
x &= f(t), \\
y &= g(t), \\
z &= h(t)
\end{align*}
\]

are called the \textbf{parametric equations} of the line.
Example 1

Find the parametric equations of the line $\ell$ through $(-2, 0, 4)$ in the direction

$$\mathbf{v} = 2\mathbf{i} + 4\mathbf{j} - 2\mathbf{k}$$

$$= (2, 4, -2).$$

Example 2

Find the parametric equations of the line $\ell$ through $P = (-3, 2, -3)$ and $Q = (1, -1, 4)$. 
### Summary

- A parametrisation of the line through a point $P$ parallel to a vector $v \neq 0$ is
  $$p + tv, \quad t \in \mathbb{R},$$
  with support vector $p = \overrightarrow{OP}$ and direction vector $v$.
- A parametrisation of the line through two points $P$ and $Q$ is
  $$p + tv, \quad t \in \mathbb{R}$$
  with support vector $p = \overrightarrow{OP}$ and direction vector $v = \overrightarrow{PQ}$.

### Warning

Parametrisations are not unique:

- Every point on the line can be chosen as support vector.
- Every non-zero vector parallel to the line can be chosen as direction vector.

### Intersection of two lines in $\mathbb{R}^3$

- Suppose two lines $\ell$ and $m$ have parametrised vector forms $p + tv$ and $q + sw$ respectively.
- An intersection is found if there are values for $t$ and $s$ such that
  $$p + tv = q + sw.$$  
  (*)
- Since vector equations in $\mathbb{R}^3$ yield three equations, equation (*) may fail to have a solution, even if $\ell$ and $m$ are not parallel.
- Non-parallel lines that do not intersect are called skew.
Example

Let \( \ell \) be the line with support vector \((-3, -3, 1)\) and direction vector \((2, 1, 1)\). Let \( m \) be the line with support vector \((2, -3, -2)\) and direction vector \((-1, 2, 4)\). Determine if \( \ell \) and \( m \) intersect, and if so, find the intersection point.

Example

Let \( \ell \) be the line with support vector \((-3, -3, 0)\) and direction vector \((2, 1, 1)\). Let \( m \) be the line with support vector \((2, -3, -2)\) and direction vector \((-1, 2, 4)\). Determine if \( \ell \) and \( m \) intersect, and if so, find the intersection point.
Planes in space

3.1 Definition

A plane in \( \mathbb{R}^3 \) is defined by an equation of the form

\[ M : ax + by + cz = d \]

with \( a, b, c \) and \( d \) real numbers.

Examples:

- The plane \( M_1 \) defined by
  \[ M_1 : x + y + z = 1 \]
  passes through the points \((1, 0, 0), (0, 1, 0)\) and \((0, 0, 1)\).

- The plane \( M_2 \) defined by
  \[ M_2 : x + y + z = 0 \]
  passes through \( O \) and is parallel to \( M_1 \).

- The plane \( M_3 \) defined by
  \[ M_3 : 2y = 3 \]
  is the plane through \((0, 3/2, 0)\) parallel to the \(xz\)-plane.

3.2 Support vector of a plane

Definition

A support vector of a plane \( M \) is a vector \( \mathbf{p} = \overrightarrow{OP} \) with \( P \) a point of \( M \).

- Suppose \( M \) is defined by \( ax + by + cz = d \), and let \( P = (x_0, y_0, z_0) \) be a point in \( M \), then \( ax_0 + by_0 + cz_0 = d \), hence
  \[ a(x - x_0) + b(y - y_0) + c(z - z_0) = 0 \]
  for all \((x, y, z)\) in \( M \).

Definition

The equation

\[ a(x - x_0) + b(y - y_0) + c(z - z_0) = 0 \]

is called the vector equation of \( M \).
**Normal vectors**

**Definition**

A **normal vector** of a plane $M$ is a vector $\mathbf{n} \neq \mathbf{0}$ that is perpendicular to $M$.

- Let $M$ be a plane defined by the vector equation
  \[
  a(x - x_0) + b(y - y_0) + c(z - z_0) = 0,
  \]
  then for all $(x, y, z)$ in $M$:
  \[
  (a, b, c) \cdot (x - x_0, y - y_0, z - z_0) = 0,
  \]
  \[
  (a, b, c) \cdot (x, y, z) - (x_0, y_0, z_0) = 0.
  \]

- Define $\mathbf{x} = (x, y, z)$, $\mathbf{p} = (x_0, y_0, z_0)$ and $\mathbf{n} = (a, b, c)$, then
  \[
  \mathbf{n} \cdot (\mathbf{x} - \mathbf{p}) = 0 \quad \text{for all } \mathbf{x} \text{ in } M.
  \]

**Definition**

The equation $\mathbf{n} \cdot (\mathbf{x} - \mathbf{p}) = 0$ is called the **normal equation** of $M$.

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**The normal equation**

**Theorem**

Let $M$ be defined by the normal equation $\mathbf{n} \cdot (\mathbf{x} - \mathbf{p}) = 0$, where $\mathbf{n}$ is a normal vector of $M$, and let $\mathbf{p} = (x_0, y_0, z_0)$ be a support vector. If $X = (x, y, z)$ is a point of $M$ then $\mathbf{n} \perp \overrightarrow{PX}$.

- Note that $\overrightarrow{PX} = \mathbf{x} - \mathbf{p}$. 
Example

Find an equation of the plane $M$ through $(-3, 0, 7)$ orthogonal to $n = (5, 2, -1)$.

Example

Find a normal equation for the plane $M: y - 2z = 4$. 
Unlike lines in $\mathbb{R}^2$, lines in $\mathbb{R}^3$ cannot be described by one equation: a linear equation $ax + by + cz = d$ describes a plane.

In order to describe a line you need two equations:

$$\begin{cases} ax + by + cz = d \\ px + qy + rz = s \end{cases}$$

Regard a line as the intersection of two planes:

Example

Give a parametrisation of the line described by the equations

$$\begin{cases} x + y - 2z = -1 \\ 2x - y + z = 2 \end{cases}$$
Example (continued)

Check your answer!
Intersection of a line and a plane

Example

The line $\ell$ is defined by the parametrisation

$$x = \frac{8}{3} + 2t, \quad y = -2t, \quad z = 1 + t, \quad t \in \mathbb{R}.$$ 

Find the intersection of $\ell$ and the plane $3x + 2y + 6z = 6$.

Application: perspective projection

- Parametrise the line $\ell$ as follows:

  $$\ell : \mathbf{r}(t) = (x_0, 0, 0) + t(x_1 - x_0, y_1, z_1), \quad t \in \mathbb{R}.$$ 

- The intersection of $\ell$ and the $yz$-plane is $P = \mathbf{r}(t_0)$ with $t_0 = \frac{x_0}{x_0 - x_1}$.

- For $P = (0, y, z)$ we have

  $$y = t_0 y_1 = \frac{x_0 y_1}{x_0 - x_1} \quad \text{and} \quad z = t_0 z_1 = \frac{x_0 z_1}{x_0 - x_1}.$$